Math 565: Functional Analysis
Lecture 9

Real Riesz representation theorem. For an lcH space X, $C_0(X,\mathbb{R})^* \cong RM_{\mathbb{R}}(X)$, more precisely, the map $\rho \mapsto I_\rho : RM_{\mathbb{R}}(X) \to C_0(X,\mathbb{R})^*$ is an isometric isomorphism.

We prove this by splitting bdd linear functionals into differences of bdd positive linear functionals. A linear functional $I : C_0(X,\mathbb{C}) \to \mathbb{C}$ is called **positive** if $I(f) \geq 0$ whenever $f \geq 0$, for all $f \in C_0(X,\mathbb{C})$; same with $\mathbb{R}$ instead of $\mathbb{C}$.

Jordan decomposition for linear functionals. Let X be any top. space. Every $I \in C_0(X,\mathbb{R})^*$ decomposes into a difference $I = I_+ - I_-$ of positive $I_+, I_- \in C_0^*(X,\mathbb{R})^*$.

Proof. For all nonnegative $f \in C_0(X,\mathbb{R})$, we define

$$I_+(f) := \sup\{I(g) : 0 \leq g \leq f\}.$$

(i) Note that $I_+(f) \geq 0$ because we can take $g := 0$, so $I(g) = 0$.

(ii) Because for each $g \in C_0(X,\mathbb{R})$ with $0 \leq g \leq f$, we have $|I(g)| \leq \|I\| \cdot \|g\|_u \leq \|I\| \cdot \|f\|_u$, we have $|I_+(f)| \leq \|I\| \|f\|_u$ so $\|I_+\| \leq \|I\|$.

(iii) For $c \geq 0$, $I_+(cf) = \sup_{0 \leq g \leq f} I(cg) = c \cdot \sup_g I(g) = c \cdot I_+(f)$.

(iv) $I_+(f_1+f_2) = I_+(f_1) + I_+(f_2)$.

Proof. $I_+(f_1+f_2) \geq I_+(f_1) + I_+(f_2)$ because if $g_i \leq f_i$ then $g_1+g_2 \leq f_1+f_2$. Conversely, if $0 \leq g \leq f_1+f_2$ then take $g_1 := \min(g, f_1) \leq g$, so $g_2 := g - g_1 \geq 0$, and get $0 \leq g_1 \leq f_1$
if $g_1(x) = g(x)$ then $g_2(x) = g(x) - g_1(x) = 0 \leq f_2(x)$; and
if $g_1(x) = f_1(x)$ then $g_2(x) = g(x) - f_1(x) \leq f_1(x) + f_2(x) - f_1(x) = f_2(x)$. Hence
$I_+(f_1) + I_+(f_2) \geq I(g_1) + I(g_2) = I(g)$, so $I_+(f_1) + I_+(f_2) \geq \sup_g I(g) = I_+(f_1+f_2)$. $\boxed{\text{iv}}$

For $f \in C_0(X,\mathbb{R})$, let $f = f_+ - f_-$ be the unique decomp. into $f_+, f_- \geq 0$, and define

$$I_+(f) := I_+(f_+) - I_+(f_-).$$

By (i), $I_+(f) \geq 0$ if $f \geq 0$, and by (iv) I_+ is additive. By (iii) for $c \geq 0$, we have
$I_+(cf) = I_+(cf_+) - I_+(cf_-) = c\,(I_+(f) - I_+(f_-)) = c I_+(f)$, and

$$I_+(-f) = I_+(-f_+ + f_-) = I_+(f_- - f_+) = I_+(f_-) - I_+(f_+) = -(I_+(f_+) - I_+(f_-)) = -I_+(f),$$

so I_+ is a positive linear functional.

Furthermore, $|I_+(f)| = |I_+(f_+) - I_+(f_-)| \overset{I_+(f_+),\, I_+(f_-) \geq 0}{\leq} \max\{I_+(f_+), I_-(f_-)\} \overset{\text{by (ii)}}{\leq} \|I\| \cdot \max\{\|f_+\|_u, \|f_-\|_u\}$
$= \|I\| \cdot \|f\|$, so $\|I_+\| \leq \|I\|$.

Now set $I_- := I_+ - I$, so $I_- \in C_0(X, \mathbb{R})^*$ and we show that I_- is positive. For $f \geq 0$,

$$I_-(f) = I_+(f) - I(f) = \sup_{0 \leq g \leq f} I(g) - I(f) = \sup_{0 \leq g \leq f} (I(g) - I(f)) = \sup_{0 \leq g \leq f} I(g - f) \geq I(f - f) = 0.$$ QED

Thus it now suffices to prove that every bdd positive linear functional on $C_0(X, \mathbb{R})$ comes comes from an unsigned Radon measure on X. We proving a slightly more general:

Positive Reisz representation theorem. Let X be an lcH space. For every positive linear functional I on $C_c(X)$, there is a unique (unsigned) Radon measure μ on X with $I = I_\mu$. Moreover, for all open $U \subseteq X$ and compact $K \subseteq X$, μ satisfies:

$$(*U) \qquad \mu(U) = \sup\{I(f) : \underbrace{0 \leq f \leq 1 \text{ and } \operatorname{supp} f \subseteq U}_{f \prec U}, f \in C_c(X)\};$$

$$(*K) \qquad \mu(K) = \inf\{I(f) : f \geq \mathbb{1}_K, f \in C_c(X)\}.$$

Remark. Every bdd linear functional on $C_c(X)$ extends uniquely to a bdd lin. functional on $\overline{C_c(X)} = C_0(X)$ by the HW question, so this statement with $C_c(X)$ is more general than it would be with $C_0(X)$. In fact, it is stricty more general as the following example shows.

Example. Let $X := \mathbb{R}$ and let $I(f) := \int f\,dx$, the Riemann integral of f. This doesn't extend to to a positive linear functional on $C_0(\mathbb{R}, \mathbb{R})$ becase then $I(f) = \infty$ for $f(x) := \begin{cases} \frac{1}{x} & \text{if } x \geq 1 \\ \max(x, 0) & \text{if } x < 1 \end{cases} \in C_0(\mathbb{R}, \mathbb{R})$ became $I(f) \geq I(g_n) \geq n$ for some uniform approximation $(g_n) \subseteq C_c(\mathbb{R}, \mathbb{R})$.
Note that $I = I_\lambda$ where λ is Lebesgue measure, so the positive Reisz representation says that one

can define Lebesgue measure solely from the knowledge of Riemann integral

Proof of uniqueness. Suppose that $I = I_\mu$ for some Radon measure μ on X. By outer regularity of μ, it suffices to show that I determines the values of μ on open sets. Thus, it's enough to show that $(*U)$ holds. To this end, note that if $f \prec U$ then $I(f) = I_\mu(f) \le I_\mu(\mathbb{1}_U) = \mu(U)$ so $\mu(U) \ge I(f)$. Conversely, the tightness of μ on U gives for each $\varepsilon > 0$ a compact set $K \subseteq U$ with $\mu(K) \approx_\varepsilon \mu(U)$ so by Urysohn we get $\mathbb{1}_K \le f \prec U$, hence $\mu(K) = I_\mu(\mathbb{1}_K) \le I_\mu(f) \le \mu(U)$, so $I_\mu(f) = I(f) \approx \mu(U)$. □

Proof of existence. For each open $U \subseteq X$, we define $\mu(U)$ by $(*U)$, i.e.

$$\mu(U) := \sup\{I(f) : f \prec U,\ f \in C_c(X)\}.$$

We then define an "outer" measure $\tilde\mu$ on all subsets $B \subseteq X$ by

$$\tilde\mu(B) := \inf\{\mu(U) : U \supseteq B \text{ open}\}.$$

Observe that for open $U \subseteq X$, $\tilde\mu(U) = \mu(U)$.

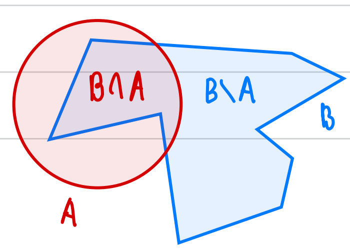
Claim 1 (key point). $\tilde\mu$ is ctbly subadditive, i.e. if $B = \bigcup_{n\in\mathbb{N}} B_n$ then $\tilde\mu(B) \le \sum_{n\in\mathbb{N}} \tilde\mu(B_n)$.

Proof. It suffices to show ctbl subadditivity for open $U = \bigcup_{n\in\mathbb{N}} U_n$, U_n open since for arbitrary $B = \bigcup_{n\in\mathbb{N}} B_n$, let $U \supseteq B$ be open and $U_n \supseteq B_n$ be open and such that $\mu(U_n) \le \tilde\mu(B_n) + \varepsilon \cdot 2^{-(n+1)}$, and also WLOG $U_n \subseteq U$, so suppose WLOG, $U = \bigcup_{n\in\mathbb{N}} U_n$. Then

$$\tilde\mu(B) \le \mu(U) \le \sum_{n\in\mathbb{N}} \mu(U_n) \le \sum_{n\in\mathbb{N}} \tilde\mu(B_n) + \sum_{n\in\mathbb{N}} \varepsilon \cdot 2^{-(n+1)} = \sum_n \tilde\mu(B_n) + \varepsilon.$$

Now for $U = \bigcup_{n\in\mathbb{N}} U_n$ all open, let $f \prec U$ and $K := \operatorname{supp} f$. So $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of K, hence by compactness, there is a finite subcover $\{U_n\}_{n<N}$. Let $(f_n)_{n<N} \subseteq C_c(X)$ be a partition of unity subordinate to $\{U_n\}_{n<N}$, i.e. $f_n \prec U_n$ for all $n<N$ and $\sum_{n<N} f_n \ge \mathbb{1}_K$. By positivity of I, I is monotone, so $f \le \mathbb{1}_K \le \sum_{n<N} f_n$ implies $I(f) \le \sum_{n<N} I(f_n) \le \sum_{n<N} \mu(U_n) \le \sum_{n\in\mathbb{N}} \mu(U_n)$, hence $\mu(U) \le \sum_{n\in\mathbb{N}} \mu(U_n)$. □

Recall that we call a set $A \subseteq X$ $\tilde\mu$-conservative (or Carathéodory measurable) if for all $B \subseteq X$, $\tilde\mu(A) = \tilde\mu(B \cap A) + \tilde\mu(B \setminus A)$.



The proof of Carathéodory's extension theorem shows that if $\tilde{\mu}$ is ctbly subadditive and $\tilde{\mu}(\emptyset)=0$, then $\tilde{\mu}$-conservative sets form a σ-algebra $\mathcal{S}$ and $\tilde{\mu}$ is a (ctbly additive) measure on $\mathcal{S}$.

<u>Claim.</u> Open sets are $\tilde{\mu}$-conservative, hence $\mathcal{S} \supseteq \mathcal{B}(X)$ so $\tilde{\mu}$ is a Borel measure on X.

Proof. Let $B \subseteq X$ and $U \subseteq X$ be open. Need to show $\tilde{\mu}(B) \geq \tilde{\mu}(B \cap U) + \tilde{\mu}(B \setminus U)$ because we already know $\leq$ by subadditivity. Let $V \supseteq B$ be open such that $\tilde{\mu}(B) \approx_\varepsilon \mu(V)$, so it's enough to show $\mu(V) \geq \mu(V \cap U) + \tilde{\mu}(V \setminus U)$ since $\mu(V\cap U) + \tilde{\mu}(V \setminus U) \geq \tilde{\mu}(B \cap U) + \tilde{\mu}(B \setminus U)$.

Let $f \prec V \cap U$ with $\mu(V \cap U) \approx_\varepsilon I(f)$. Then putting $K := \operatorname{supp} f$, $V \setminus K$ is open and $\mu(V \setminus K) \geq \tilde{\mu}(V \setminus U)$. Let $g \prec V \setminus K$ with $\mu(V \setminus K) \approx_\varepsilon I(g)$, and $f+g \prec V$, hence

$$\mu(V) \geq I(f+g) = I(f) + I(g) \approx_{2\varepsilon} \mu(V \cap U) + \mu(V \setminus K) \geq \mu(V \cap U) + \tilde{\mu}(V \setminus U). \quad \square$$

Denote $\mu := \tilde{\mu}|_{\mathcal{B}(X)}$, so μ is a Borel measure. By definition, μ satisfies $(*U)$, and is outer regular by the definition of $\tilde{\mu}$. Furthermore, $(*K)$ holds by approximating a given compact $K \subseteq X$ with an open $U \supseteq K$ and using Urysohn to get an $f \in C_c(X)$ with $\mathbb{1}_K \leq f \prec U$. $(*U)$ and $(*K)$ together also imply that μ is tight on open sets and finite on compact sets, hence a Radon measure.

It remains to prove that $I = I_\mu$, for which it is enough to show $I(f) = I_\mu(f)$ for all $f \in C_c(X, [0,1])$ since this set linearly spans $C_c(X, \mathbb{C})$. One does this by taking a layered cake decomposition of f (similar to the proof that every measurable $g \geq 0$ is an increasing pointwise limit of simple functions), $f = \sum_{i \leq n} f_i$, and showing that both $I(f_i)$ and $I_\mu(f_i)$ are tightly sandwiched between the measures of the ith and $(i-1)^{\text{th}}$ layers, hence they must coincide since the error is arbitrarily small.

$\boxed{\text{QED}}$